

Some exact solutions with torsion in 5-D Einstein-Gauss-Bonnet gravity

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Abstract

Exact solutions with torsion in Einstein-Gauss-Bonnet gravity are derived. These solutions have a cross product structure of two constant curvature manifolds. The equations of motion give a relation for the coupling constants of the theory in order to have solutions with nontrivial torsion. This relation is not the Chern-Simons combination. One of the solutions has a $AdS_2 \times S^3$ structure and is so the purely gravitational analogue of the Bertotti-Robinson space-time where the torsion can be seen as the dual of the covariantly constant electromagnetic field.

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1 Introduction

It is a well known fact that in four dimensions the Einstein-Hilbert action (plus a cosmological term) is the only functional that can be built out of the curvature invariants leading to second order field equations. In fact higher

order terms in the curvature invariants lead generically to higher order field equations which at quantum level would lead to ghosts spoiling the unitarity of the theory[1]. The Einstein Hilbert action can however be generalized in a straightforward way to higher dimensions. Indeed there exist large class of theories containing higher powers in the curvature which lead to second order equations for the metric, known as Lovelock theories [2].

Now in the Einstein-Hilbert action, the vielbein e^a and spin connection ω^{ab} can be treated as independent fields. This is known as the first order formalism since the field equations involve only first derivatives: such a formalism is mandatory when dealing with Fermionic fields. One of the characteristic features of the Einstein-Hilbert action (in n dimensions) in the first order formalism is that its variation with respect to the spin connection gives equations of motion of the form

$$\epsilon_{a_1 \dots a_n} T^{a_1} \wedge e^{a_2} \wedge \dots \wedge e^{a_{n-2}} = 0$$

which simply imply the vanishing of the torsion two-form $T^a = 0$. However in dimension higher than four the Einstein-Hilbert action is no longer the unique possible first order action. In fact, the Lovelock theories also admit a first order formulation [3]¹. In five dimensions for example one can add to the standard Einstein-Hilbert action with a cosmological constant an extra term to the usual action known as the Gauss-Bonnet term which in familiar notation reads

$$\int d^5x \sqrt{g} (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}).$$

Above $R_{\mu\nu\rho\sigma}$ is the Riemann tensor, $R_{\mu\nu}$ the Ricci tensor and R the Ricci scalar. We shall employ the differential form notation. Introducing the curvature two-form, R^{ab} , the above term is equal to

$$\int R^{ab} \wedge R^{cd} \wedge e^e \epsilon_{abcde}.$$

Besides being a natural generalization of General Relativity to the five dimensional case, Einstein-Gauss-Bonnet theories appear to be quite compatible with the available astrophysical and cosmological experimental data (see, for

¹As well as the Lovelock terms, one can also add to the action terms explicitly involving the torsion and Lorentz Chern-Simons terms related to the Pontryagin form [4]. However, in this paper we focus on five dimensions, where no such terms exist.

an incomplete list of references, [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]). The addition of Lovelock terms to the action affects the equations of motion in such a way that they no longer imply that the torsion vanishes. Instead, it becomes a new propagating degree of freedom. The presence of torsion could have interesting phenomenological consequence (see, for instance, Ref. [17]). Torsion has a deep geometrical meaning which could shed some light on non-perturbative features of gravitational theories which cannot be taken into account in the standard formalism.

Generally the Lovelock equations of motion, combined with the Bianchi identities, give very strong constraints on the torsion [18]. In most cases one obtains an over-determined system of equations, making it extremely difficult to find exact solutions with non-vanishing torsion. There is a special case where solutions with torsion are known: in odd dimensions, for a certain tuning of the coupling constants, the Lovelock theory becomes equivalent to a Chern-Simons theory [19]. Such theories have an enlarged local symmetry group which allows (roughly speaking) to fit inside a “bigger curvature” for the AdS group both the standard curvature and the torsion providing one with the needed mathematical structures to formulate a supersymmetric theory as well. Because of such local symmetry, the field equations are suitable to investigate non trivial configurations such as black holes and worm holes (see, for instance, [20]). This combination of coefficients is unique in that the field equations do not place strong constraints on the torsion. A black hole with torsion was found in Ref. [21]. In that case, due to the enhanced gauge symmetry of the Chern-Simons theory, the solution was related to a torsion-free solution by a gauge transformation. Other solutions with torsion in Chern-Simons gravity are given in Refs. [22].

In this paper we will exclude the Chern-Simons combination. The goal of this paper will be to present some exact solutions with non-vanishing torsion in five dimensional Lovelock gravity in the non-Chern-Simons case. It seems that until now, no explicit solutions have been considered in the literature (In Ref. [23] there is a nice general discussion of spherically symmetric torsion, where it was shown that static black holes in higher dimensional Lovelock gravity generically have zero torsion). In order to overcome the previously explained difficulties we use an ansatz of a space-time being the cross product of the form $N_2 \times M_3$ where both submanifolds are of constant curvature and M_3 is spacelike. The usefulness of this ansatz is due to the fact that one can search for a torsion with components only in M_3 . Because this submanifold is three dimensional, a totally anti-symmetric torsion tensor $T_{ijk} \propto \epsilon_{ijk}$ will

respect the symmetry. Note also that such a torsion is proportional to the Hodge dual of the curvature two form on M_3 . In this sense, the ansatz is inspired by an analogy with BPS states in gauge theory, as will be explained later. The equations of motion impose a relation between coupling constants which is *not* the Chern-Simons combination.

One of the possible solutions is $AdS_2 \times S_3$ which is analogous to the Bertotti-Robinson solution [24] where the role of the electromagnetic field is taken by a covariantly constant torsion. The analogy to a BPS state is clear here as the Bertotti-Robinson solution is indeed BPS. Because of this analogy one may wonder if also the solution presented here is a BPS state. However there is no obvious way to write a Killing spinor equation because, to the authors knowledge, the supersymmetric extension of the theory is not known (although some features of BPS states are indeed present as will be explained later). Anyway this BPS analogy proved to be extremely useful to find nontrivial solutions with torsion, which up to now has been an extremely difficult task.

It will be shown that this solution has no zero torsion limit. On the other hand it is easy to see that the torsion free $AdS_2 \times S^3$ is a solution for Einstein-Gauss-Bonnet gravity. The new solution presented here seems to be therefore a topological excitation. It will be shown that also other solutions with the cross product structure of constant curvature manifolds exist.

The structure of the paper will be as follows: In section 2 the Einstein-Gauss-Bonnet theory with torsion is reviewed. In section 3, the new solutions are described. In section 4, the analogy with BPS states is developed and some interesting features of the solutions are investigated. Section 5 is a summary of the main conclusions.

2 Einstein-Gauss-Bonnet theory with torsion

2.1 Gravity with Torsion

Since the Kaluza-Klein idea and with the advent of string theories the possibilities to have extra dimensions comes into play.

As mentioned previously, the Lovelock Lagrangian is the natural extension of GR to higher dimensions. Let us briefly review Lovelock gravity in first order formalism, the relation with the second order formalism will be described in the next subsection in the five dimensional case. The action has

the form (for a detailed review, see Ref. [25]):

$$I_D = \kappa \int \sum_{p=0}^{[D/2]} \alpha_p L^{(D,p)},$$

$$L^{(D,p)} \equiv \varepsilon_{a_1 \dots a_D} R^{a_1 a_2} \wedge \dots \wedge R^{a_{2p-1} a_{2p}} \wedge e^{a_{2p+1}} \wedge \dots \wedge e^{a_D}.$$

Here $e^a = e_\mu^a dx^\mu$ is the vielbein, $\omega_b^a = \omega_{b\mu}^a dx^\mu$ the spin connection, η_{ab} the Minkowski metric in the vielbein indices, $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$ is the spacetime metric. The curvature and torsion are:

$$R_b^a \equiv (d\omega + \omega \wedge \omega)_b^a,$$

$$T^a \equiv De^a = de^a + \omega_b^a \wedge e^b.$$

It is manifest in this way that the vielbein indices a, b, \dots behave as internal gauge indices. The Bianchi identities read

$$DR_b^a = dR_b^a + \omega_c^a \wedge R_b^c + \omega_b^c \wedge R_c^a = 0,$$

$$DT^a = R_b^a \wedge e^b.$$

In $D = 4$ the above tells that one is free to add a further term (the so called Gauss-Bonnet term) to S_{EH} which, being a topological invariant, does not change the Euler-Lagrange equations. In higher dimensions the situation changes: extremizing the Lovelock Lagrangian of order p one obtains the following types of equations:

$$\sum_{p=0}^{[D/2]} (D - 2p) \alpha_p \Xi_a^{(p)} = 0, \quad (1)$$

$$\sum_{p=0}^{[D/2]} p (D - 2p) \alpha_p \Xi_{ab}^{(p)} = 0, \quad (2)$$

where

$$\Xi_a^{(p)} \equiv \varepsilon_{ab_2 \dots b_D} R^{b_2 b_3} \dots R^{b_{2p} b_{2p+1}} e^{b_{2p+2}} \dots e^{b_D}, \quad (3)$$

$$\Xi_{ab}^{(p)} \equiv \varepsilon_{aba_3 \dots a_D} R^{a_3 a_4} \dots R^{a_{2p-1} a_{2p}} T^{a_{2p+1}} e^{a_{2p+2}} \dots e^{a_D}. \quad (4)$$

In four dimension S_{EH} is basically the only first order action². In this case the torsion plays no role, at least classically. In higher dimensions the torsion emerges as a natural geometrical object.

2.2 Five dimensional case

In this paper, we consider the five dimensional Einstein-Gauss-Bonnet action which in the familiar formalism reads:

$$I = \kappa \int d^5x \sqrt{g} (R - 2\Lambda + \alpha (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})) , \quad (5)$$

where κ is related to the Newton constant, Λ to the cosmological term, and α is the Gauss-Bonnet coupling. For later convenience, it is useful to express the action (5) in terms of differential forms as³

$$I = \int \left(\frac{c_0}{5} e^a e^b e^c e^d e^e + \frac{c_1}{3} R^{ab} e^c e^d e^e + c_2 R^{ab} R^{cd} e^e \right) \epsilon_{abcde} \quad (6)$$

where, as explained in the previous section, $e^a = e^a_\mu dx^\mu$ is the vielbein, and $R^{ab} = d\omega^{ab} + \omega^a_f \omega^{fb}$ is the curvature 2-form for the spin connection $\omega^{ab} = \omega^{ab}_\mu dx^\mu$. The equations of motion obtained by varying the action with respect to the spin connection ω^{ab} read

$$\mathcal{E}_{ab} \equiv T^c (c_1 e^d e^e + 2c_2 R^{de}) \epsilon_{abcde} = 0 \quad (7)$$

The equations of motion obtained varying the action with respect to the vielbein e^a read

$$\mathcal{E}_e \equiv (c_0 e^a e^b e^c e^d + c_1 R^{ab} e^c e^d + c_2 R^{ab} R^{cd}) \epsilon_{abcde} = 0. \quad (8)$$

When the coefficients of the theory satisfy a special fine-tuning the above action turns out to be of a Chern-Simons theory[19]. In this section the case of Chern-Simons will be explicitly excluded by imposing the inequality:

$$c_1^2 \neq 4c_0 c_2 . \quad (9)$$

²It should be noted that there are more general actions with torsion that can be constructed if one does not insist on a first order theory. In four dimension the torsion plays an important geometrical role since the important topological invariant (constructed by Nieh and Yan) can be constructed $N = T^a T_a - e^a e^b R_{ab}$. Such an invariant appears in the anomalous term of the divergence of the chiral anomaly [26].

³The relationship between the constants appearing in Eqs (5) and (6) is given by $\alpha = \frac{c_2}{2c_1}$, $\Lambda = -6\frac{c_0}{c_1}$, $\kappa = 2c_1$.

or, equivalently

$$\frac{4\alpha\Lambda}{3} \neq -1.$$

3 Exact solutions with torsion

The main idea is that an ansatz, inspired by BPS states in field theory, could allow one to find non trivial solutions with non vanishing torsion. As BPS states in field theory have non trivial topological charges, in the present case such *vacuum solutions* of Einstein-Gauss-Bonnet gravity could manifest some properties of solutions in the presence of matter fields carrying some charges. For a suitable choice of the coefficients which *is not* Chern-Simons, a set of solutions will be constructed.

One of these solutions is the vacuum analogue of the Bertotti-Robinson metric in which the torsion plays the role of the electromagnetic field. This will be derived in the next subsection before proceeding to the more general solutions.

3.1 $AdS_2 \times S_3$ solution

We search for a $AdS_2 \times S_3$ solution with torsion. The idea is to find the analogous of a Bertotti-Robinson metric in which the torsion plays the role of the electromagnetic field. Therefore the following ansatz for the metric is natural

$$ds^2 = \frac{l^2}{x^2} (-dt^2 + dx^2) + \frac{r_0^2}{4} (d\phi^2 + d\theta^2 + d\psi^2 + 2 \cos \theta d\phi d\psi) . \quad (10)$$

As vielbein we choose:

$$e^0 = \frac{l}{x} dt \quad ; \quad e^1 = \frac{l}{x} dx \quad ; \quad e^i = r_0 \tilde{e}^i \quad (11)$$

where r_0 , the radius of the 3-sphere, is a constant and \tilde{e}^i is the intrinsic vielbein on the unit sphere. For definiteness, the Poincaré coordinates have been used for the two dimensional Anti de Sitter space and Euler angles for the sphere.

We make the following ansatz for the torsion, consistent with the spherical symmetry:

$$T^1 = 0 \quad ; \quad T^0 = 0 \quad ; \quad T_i = \frac{H}{r_0} e^j e^k \epsilon_{ijk} , \quad (12)$$

where H is a constant.

Note that on the unit three-sphere there exists a choice of intrinsic vielbein such that $\tilde{\omega}^{ij} = -\epsilon^{ijk}\tilde{e}_k$, where $\tilde{\omega}^{ij}$ is the intrinsic Levi-Civita spin connection. With this choice, the 5-dimensional spin connection reads

$$\omega^{01} = -\frac{1}{x}dt, \quad \omega^{ij} = (H+1)\tilde{\omega}^{ij} = -(H+1)\epsilon^{ijk}\tilde{e}_k. \quad (13)$$

Thanks to the geometric structure of the sphere, the torsion can be written in a way that is homogeneous and isotropic thanks to the invariance of the tensor ϵ^{ijk} on the sphere. The naturalness of this ansatz will be discussed more in section 4. The curvature turns out to be

$$R^{01} = -\frac{1}{l^2}e^0e^1; \quad R^{ij} = \frac{1-H^2}{r_0^2}e^ie^j \quad (14)$$

One recovers the torsionless case setting $H = 0$. Inserting Eqs. (11), (12) and (14) in the equations of motion one obtains from the (ij) component of (7):

$$c_1 - \frac{2c_2}{l^2} = 0 \quad \text{or} \quad H = 0. \quad (15)$$

The other components of equation (7) are automatically satisfied. From the (0) and (1) components of eq. (8) we get

$$4c_0 + 2c_1 \frac{(1-H^2)}{r_0^2} = 0 \quad (16)$$

The (i) component of eq. (8) gives

$$12c_0 + 2c_1 \left(-\frac{1}{l^2} + \frac{1-H^2}{r_0^2} \right) - \frac{4c_2}{l^2} \frac{(1-H^2)}{r_0^2} = 0 \quad (17)$$

It is worth stressing here that the form of the torsion of Eq. (12) greatly simplify the equations of motions (which in the Einstein-Gauss-Bonnet are quite complicated). In particular, some key identities have been used in deriving the above equations. The first one is that expressions like

$$\epsilon_{ijk}T^ie^je^k = 0 \quad (18)$$

identically vanish due to the fact that the torsion contains always two angular vielbeins so that such exterior products are zero because there are only three independent angular vielbeins. The second identity is that expressions like

$$\epsilon_{ijk}T^ie^je^0 \approx \epsilon_{ijk}(\epsilon^{imn}e_me_n)e^je^0 = (\delta_j^m\delta_k^n - \delta_k^m\delta_j^n)e_me_ne^je^0 = 0 \quad (19)$$

also vanish since there is always a wedge product of an angular vielbein with itself.

Substituting equations (15) and (16) in (17) one finds that there exist solutions with torsion only if the coupling constant satisfy the following relation

$$c_1^2 = 12c_0c_2 \quad (20)$$

Note that this is *not* the Chern-Simons combination of the coupling constants. The AdS_2 length scale l is completely determined by the coupling coefficients:

$$\frac{1}{l^2} = \frac{c_1}{2c_2}, \quad (21)$$

The sphere radius r_0 and the torsion parameter H are related by

$$1 - H^2 = -\frac{2c_0}{c_1}r_0^2 \quad \Rightarrow \quad H^2 = 1 + \frac{r_0^2}{3l^2}.$$

In summary, the space-time $AdS_2 \times S_3$ with vielbein given by (11) and with torsion and curvature

$$T_i = \pm \sqrt{\frac{1}{r_0^2} + \frac{1}{3l^2}} \epsilon_{ijk} e^j e^k, \quad (22)$$

$$R^{01} = -\frac{1}{l^2} e^0 e^1, \quad R^{ij} = -\frac{1}{3l^2} e^i e^j, \quad (23)$$

with l given by (21), is a solution of the Einstein-Gauss-Bonnet theory, provided that the relation (20) among the coupling constants holds, with c_2/c_1 and c_0/c_1 positive.

The torsion is bounded from below for finite sized sphere and AdS length scale. This means that there is no continuous zero torsion limit.

Moreover, the torsion is fully antisymmetric. This allows an intriguing analogy with gravity in the presence of a constant electromagnetic field. It is natural to define a three form $T \equiv T_{ijk} e^i \wedge e^j \wedge e^k = 3! \frac{H}{r_0} e^2 \wedge e^3 \wedge e^4$. Also we define the Hodge dual, the two form $*T = -3! \frac{H}{r_0} e^0 \wedge e^1$. Due to the Bianchi identities, these are covariantly constant.

$$DT = 0, \quad D * T = 0.$$

The analogy with electromagnetic field is made by defining $F \equiv *T$. Thus F is seen to obey the source-free Maxwell equations, making manifest the close resemblance with the electromagnetic field of the Bertotti-Robinson solution.

3.2 Product of two constant curvature manifolds with torsion

In the previous section a product manifold was considered. It was seen that the torsion could be introduced on the three-sphere in a way that was consistent with spherical symmetry. Furthermore such a torsion satisfied the equations of motion, provided that there was a rather curious relation (20) between the coupling constants in the action. A key feature of that solution was that the torsion tensor $T_{ijk} \propto \epsilon_{ijk}$ is manifestly of a form which is homogeneous and isotropic in the three-dimensional subspace (in that case a sphere). It is natural to generalise this solution to more general product manifolds involving a three dimensional manifold with constant curvature, which could be positive, negative or zero. In this section we shall study solutions whose metric is the direct product of two manifolds of constant Riemannian curvature, with torsion living in a three-dimensional manifold.

Let N_2 be a two-dimensional manifold with Minkowskian signature and constant curvature. The metric is given by $ds_N^2 = -e^0 \otimes e^0 + e^1 \otimes e^1$, where e^0 and e^1 are the vielbeins with Levi-Civita connection $\hat{\omega}^{01}$ and curvature satisfying

$$\hat{R}^{01} = \Lambda_N e^0 \wedge e^1. \quad (24)$$

Let M_3 be a three-dimensional manifold of constant curvature with Euclidean signature. The metric is $ds_M^2 = \delta_{ij} e^i \otimes e^j$. The three-manifold has a Levi-Civita connection⁴ $\hat{\omega}^{ij}$ and corresponding curvature \hat{R}^{ij} satisfying

$$\hat{R}^{ij} = \Lambda_M e^i \wedge e^j. \quad (25)$$

The five-dimensional spacetime will be a product space $N_2 \times M_3$, the metric being

$$ds^2 = ds_N^2 + ds_M^2. \quad (26)$$

The torsion is introduced onto the M_3 in such a way as to respect the symmetry. This is guaranteed by the invariance property of the alternating tensor ϵ_{ijk} . Let us make the ansatz:

$$T_i = \tau \epsilon_{ijk} e^j \wedge e^k. \quad (27)$$

⁴Since torsion shall be later introduced on M_3 , the Levi-Civita connection shall be denoted by $\hat{\omega}^{ij}$. The symbol ω^{ij} is reserved for the full connection including the contorsion.

We shall further assume that τ is constant, which implies that T^i is covariantly constant with respect to the Levi-Civita connection. In the appendix a more general ansatz is analysed where the symmetry of N_2 is relaxed and it is shown that the only solution is the one discussed here.

The spin connection on M_3 now takes the form $\hat{\omega}^{ij} + K^{ij}$ where K^{ij} is the contorsion 1-form. According to (27) the contorsion is:

$$K_{ij} = -\tau \epsilon_{ijk} e^k. \quad (28)$$

Let R^{ab} denote the five-dimensional curvature tensor. It is a sum of the torsion-free curvature, with components (24) and (25), and a part which comes from the torsion. This can conveniently be obtained by expanding $R^{ab} = d(\hat{\omega}^{ab} + K^{ab}) + (\hat{\omega}^a_c + K^a_c) \wedge (\hat{\omega}^{cb} + K^{cb})$ to give the well-known formula:

$$R^{ab} = \hat{R}^{ab} + \hat{D}K^{ab} + K^a_c \wedge K^{cb}. \quad (29)$$

From (28) it can be seen that the contorsion is covariantly constant with respect to the Levi-Civita connection, $\hat{D}K^{ij} = \kappa (\epsilon_{ljk} \hat{\omega}^l_i + \epsilon_{ilk} \hat{\omega}^l_j + \epsilon_{ijl} \hat{\omega}^l_k) \wedge e^k = 0$. The non-vanishing components of the curvature are

$$R^{01} = \Lambda_N e^0 \wedge e^1, \quad R^{ij} = (\Lambda_M - \tau^2) e^i \wedge e^j. \quad (30)$$

The effect of the homogeneous and isotropic torsion is to rescale the three-dimensional part of the curvature.

Now it remains to check that the field equations are satisfied by torsion (27) and curvature (30). First let us check equation (7) coming from the variation w.r.t. the spin connection. Since $\epsilon_{ijk} T^j \wedge e^k = 0$ the only non-trivial component is:

$$0 = \mathcal{E}_{ij} = 2\epsilon_{ijk} T^k \wedge (c_1 e^0 \wedge e^1 + 2c_2 R^{01}). \quad (31)$$

Now we check equation (8) coming from the variation w.r.t. the vielbein.

$$0 = \mathcal{E}_0 = \epsilon_{ijk} (4c_0 e^1 \wedge e^i \wedge e^j \wedge e^k + 2c_1 e^1 \wedge e^i \wedge R^{jk}), \quad (32)$$

$$0 = \mathcal{E}_1 = \epsilon_{ijk} (4c_0 e^0 \wedge e^i \wedge e^j \wedge e^k + 2c_1 e^0 \wedge e^i \wedge R^{jk}), \quad (33)$$

$$0 = \mathcal{E}_i = \epsilon_{ijk} \left(12c_0 e^0 \wedge e^1 \wedge e^j \wedge e^k + 2c_1 (R^{01} \wedge e^j \wedge e^k + e^0 \wedge e^1 \wedge R^{jk}) + 4c_2 R^{01} \wedge R^{ij} \right). \quad (34)$$

Equation (31) implies:

$$\Lambda_N = -\frac{c_1}{2c_2}. \quad (35)$$

Equations (32) and (33) imply

$$\Lambda_M - \tau^2 + \frac{2c_0}{c_1} = 0. \quad (36)$$

Substituting these two equations in (34) gives the relation between the coupling constants $c_1^2 = 12c_0c_2$.

3.3 Summary of the solutions

We have found solutions for the special class of Lovelock theories satisfying⁵ $c_1^2 = 12c_0c_2$. Since the possibility of a vanishing Einstein-Hilbert term is excluded, we may normalise $c_1 = 1$ without loss of generality. The action is thus:

$$I = \int \left(\frac{1}{60c_2} e^a e^b e^c e^d e^e + \frac{1}{3} R^{ab} e^c e^d e^e + c_2 R^{ab} R^{cd} e^e \right) \epsilon_{abcde}. \quad (37)$$

The metric is the product of $N_2 \times M_3$ with Riemannian curvature:

$$\hat{R}^{01} = -\frac{1}{2c_2} e^0 \wedge e^1, \quad \hat{R}^{ij} = \Lambda_M e^i \wedge e^j. \quad (38)$$

The curvature and torsion are:

$$R^{01} = -\frac{1}{2c_2} e^0 \wedge e^1, \quad R^{ij} = -\frac{1}{6c_2} e^i \wedge e^j, \quad (39)$$

$$T_i = \pm \sqrt{\Lambda_M + \frac{1}{6c_2}} \epsilon_{ijk} e^j \wedge e^k. \quad (40)$$

We see that the full non-Riemannian curvature is completely determined by the coupling constant c_2 . There is just one constant of integration Λ_M , which characterizes both the Riemannian curvature of M_3 and the torsion.

⁵In terms of the notation of equation (5), the coefficients satisfy $4\alpha\Lambda = -1$.

$\mathbb{R}_2 \times M_3$		No solution	
$\text{AdS}_2 \times S_3$	$c_2 > 0$	Λ_M arbitrary	No zero torsion limit
$\text{AdS}_2 \times H_3$	$c_2 > 0$	$-1/6c_2 \leq \Lambda_M < 0$	Zero torsion limit
$\text{AdS}_2 \times \mathbb{R}_3$	$c_2 > 0$	$\Lambda_M = 0$	No zero torsion limit
$\text{dS}_2 \times S_3$	$c_2 < 0$	$1/6 c_2 \leq \Lambda_M$	Zero torsion limit
$\text{dS}_2 \times H_3$	$(c_2 < 0)$	No solution	
$\text{dS}_2 \times \mathbb{R}_3$	$(c_2 < 0)$	No solution	

Note that the generalisation to the case that N_2 is Euclidean and with M_3 Lorentzian is straightforward.

The field equations for Einstein-Gauss-Bonnet can also be written as follows:

$$T^c \left(R^{de} + \frac{(\Lambda_+ + \Lambda_-)}{2} e^d e^e \right) \epsilon_{abcde} = 0 \quad (41)$$

$$(R^{ab} + \Lambda_+ e^a e^b) (R^{cd} + \Lambda_- e^c e^d) \epsilon_{abcde} = 0 \quad (42)$$

so that, assuming that the torsion vanishes there are two possible (A)dS vacua with different cosmological constant. Here we have found a third kind of solution with a high degree of symmetry: the product of maximally symmetric spaces with nonzero torsion. In the solutions we have found above, it is the average of the two cosmological constants which is important,

$$\pm \frac{1}{l^2} = \frac{(\Lambda_+ + \Lambda_-)}{2},$$

because it determines the cosmological constant of the N_2 .

4 Field theoretical features in first-order gravitational theories

In Ref. [27] some analogies were investigated between BPS states in field theory on the one hand and Gravitational theories with torsion on the other. Let us briefly revisit this subject in the light of the solutions found in section 2. For detailed reviews on BPS states in field theory see Refs. [28, 29]. We have focused on the Lovelock theories because they have a first order formalism. It is not a scope of the present paper to analyze all the possible higher curvature corrections (which are expected on various theoretical grounds ranging

from string theory to Kaluza-Klein reductions) to standard Einstein-Hilbert action, because such corrections are higher order in derivatives. A detailed review on how generic higher curvature corrections may arise and on their interesting physical effects can be found in [31] and references therein. On the other hand, the formal analogy pointed out in [27] is only based on the geometrical roles of the Higgs field and the gauge connection on one hand and of the vielbein and the spin connection on the other hand while the detailed form of the field equations is not so important. Therefore, the present approach to analyze the dynamical role of torsion could also work in some more general cases not belonging to the Lovelock class.

In the Yang-Mills-Higgs theory, the BPS equations involve typically linear relations (such as higher dimensional self-duality conditions) among $D^a\phi$ [the covariant derivative of the Higgs field] and F^{ab} [the Yang-Mills field strength] in which ϕ can enter quadratically (as, for instance, in the vortex case[28, 29]). Inspired by this, a natural ansatz for gravity with torsion is the linear relation

$$T^c = f_{ab}^c (\alpha R^{ab} + \beta e^a e^b) \quad (43)$$

where f_{ab}^c is an appropriately chosen three index tensor and α and β are two constants.

Now there does not exist a genuine invariant tensor with three indices (that is, f_{ab}^c in Eq. (43)) in the five-dimensional Lorentz group. A solution involving such a tensor would necessarily break some of the Lorentz symmetry. One of the features of topological defects is precisely that they partially break Lorentz invariance (the surviving generators are the ones leaving invariant the defects). For instance, when in quantum field theory one expands around the trivial vacuum (all the fields equal to zero) the Lorentz generators annihilate the vacuum (see, for instance, [30]). When expanding around non trivial saddle points (that is, topological defects) this is not so since the position and the structure of the topological defects make the action of the Lorentz generators on the vacuum non trivial.

In the case of our solutions, there is a three-dimensional submanifold of constant curvature. Thus it is natural to choose the tensor $f_{ijk} = \epsilon_{ijk}$ consistent with the unbroken Lorentz generators.

Since our solutions are of the form (43), there is some analogy between our solutions and BPS states. However, because of the very different structure of Lovelock action, it is not easy to make this analogy precise. It is not easy to construct an energy functional from which to deduce a BPS bound. In

spite of the fact that the supergravity or a BPS bound are not known, some interesting features of BPS states can be analyzed without the tools of SUSY. In particular, in the vacuum analogue of the Bertotti-Robinson it appears that the torsion is bounded from below for finite sized sphere and AdS length scale. This means that there is no continuous zero torsion limit. So that such a solution appears as a topological excitation in which intrinsically the torsion *cannot be small*. Another interesting feature is related to the rigidity of the solutions due to the presence of torsion. In the non Chern-Simons case, the equations of motions have to fulfil a strong compatibility condition: taking the covariant derivative of Eq. (8) and comparing with Eq. (7) it turns out that [18]

$$\begin{aligned} e^b R^{cd} T^e \epsilon_{abcde} &= 0, \\ e^b e^c e^d T^e \epsilon_{abcde} &= 0. \end{aligned}$$

Such conditions are absent in the Chern-Simons case because of the tuning of the coefficients. The rigidity of the above conditions makes clear why it is so difficult to find exact solutions with intrinsic torsion (when the torsion is absent the above conditions are trivial). It is then apparent that the “BPS-inspired” ansatz for the torsion proposed in [27] is quite good because it naturally provides one with a method to solve the above conditions due to the identities (18) and (19).

As further evidence that the solution may be a topologically non-trivial vacuum, it is worth pointing out that the action (37), evaluated on the solutions is zero (compare with Euclidean instantons which have finite action), as can be easily checked. In contrast, the action evaluated on the two AdS_5 solutions, does not vanish.

5 Conclusions and outlook

It has been shown that in five dimensional Einstein-Gauss-Bonnet gravity there are exact solutions with non vanishing intrinsic torsion. These were found for a special choice of coupling constants given by the action (37), which *is not the Chern-Simons combination*. To the best of the authors knowledge, no solutions with torsion in non Chern-Simons Einstein-Gauss-Bonnet gravity have been found before. The analogies with field theory and the peculiar features of such states have been discussed. Among the solutions found there is an analogue of the Bertotti-Robinson metric in which the

torsion is the dual of a covariantly constant electromagnetic field. However, without knowing the supersymmetric extension of this theory, there is no obvious way to write the Killing spinor equation. This solution is, in a sense a topological excitation in which there is no continuous zero torsion limit. Because of the rigidity of the Einstein-Gauss-Bonnet equations of motions in the presence of nontrivial torsion there are strong constraints on small excitations (besides the re-scaling of the physical parameters appearing in the solutions). It is reasonable to expect that, for this reason, the above exact solutions with the “BPS-inspired” torsion could be a topologically non-trivial vacuum state.

In view of the analogy with the Bertotti-Robinson solution, one may also expect the existence of a solution analogue of the extreme AdS-Reissner-Nordstrom black hole which interpolates between the Bertotti-Robinson and the AdS metric. This is an open question for future research.

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A On the generality of the ansatz

Let us now argue that the solutions found in the previous section are quite general for a product metric involving a maximally symmetric three-manifold M_3 . Let us now assume that the metric is $N_2 \times M_3$, where now N_2 is an arbitrary two-manifold. That is to say, we shall not specify the form of \hat{R}^{01} .

We shall keep the same ansatz for the torsion on M_3 except that, since N_2 need not be maximally symmetric, τ may depend on (x, t) , the co-ordinates

of N_2 . Thus we look for solutions of the form:

$$T^0 = F(x, t) e^0 \wedge e^1, \quad T^1 = G(x, t) e^0 \wedge e^1, \quad T_i = \tau(x, t) \epsilon_{ijk} e^j \wedge e^k. \quad (44)$$

The contorsion is:

$$K^{01} = F e^0 - G e^1, \quad K_{ij} = -\tau \epsilon_{ijk} e^k,$$

The components of curvature along M_3 are found using the formula (29).

$$R^{ij} = (\Lambda_M - \tau^2) e^i e^j - d\tau \epsilon^{ijk} e_k. \quad (45)$$

Let us study the field equations (7) and (8). The component $\mathcal{E}_{01} = 0$ tells us immediately that $d\tau$ vanishes:

$$\tau = \text{constant}. \quad (46)$$

The components $\mathcal{E}_{0i} = 0$ and $\mathcal{E}_{1i} = 0$ imply:

$$1 - \tau^2 + \frac{c_1}{2c_2} = 0 \quad \text{or} \quad F = 0, \quad G = 0. \quad (47)$$

The component $\mathcal{E}_0 = 0$ gives:

$$1 - \tau^2 + \frac{2c_0}{c_1} = 0. \quad (48)$$

Comparing equations (47) and (48) we have either $4c_0c_2 = c_1^2$ or $F = G = 0$. The first of these alternatives is precisely the Chern-Simons combination. So we conclude that, for $4c_0c_2 \neq c_1^2$, we have $F = G = 0$ and $\tau = \text{constant}$.

Finally, $\mathcal{E}_{ij} = 0$ imposes that the curvature of N_2 is constant. Thus the solution reduces to that of the previous section.

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